Math as Flexibility of Mind

Volume II: Roses

1 Course on Transfinite Set Theory

1.1 Day 1 (Week 1)

1.1.1 A Brief Introduction to Transfinite Set Theory

Write on the board

"Transfinite Set Theory"

What does the word "transfinite" mean? Start with "trans-". What is infinity? God? Is the universe infinite? What is the current theory? Discuss.

In the same way that James Brown is the godfather of soul, Georg Cantor is the godfather of transfinite set theory. Discuss Cantor's struggles (there is a nice BBC documentary on the subject - Dangerous Knowledge).

"To see a universe in a grain of sand
and heaven in a wildflower
to hold infinity in the palm of your hand
and eternity in an hour."
-William Blake Auguries of Innocence

Discuss. What do these lines have to do with what we will be studying? Can anyone recite by memory? In front of the whole class? We read our loud to learn deeply/review.

Note that if you can hold something in the palm of your hand, what is it? A noun! Not a verb. This was hard for Cantor's contemporaries to understand, he was too far ahead of them.

Definition 1: A set is a collection of objects/elements. In Cantor's words: "A many which can be thought of as a one."

Examples: $\mathcal{S} = \{$ All students in FOAM $\}$. Can I say that David belongs to/is a member of $\mathcal{S}$? If so, I would write

$$D \in \mathcal{S}$$

$$\mathcal{D} = \{\text{ All desks in the room }\}$$

Remember, we can name things whatever we want. So is Ralph $\in \mathcal{D}$?

Course created by Leonard Boehm and recorded by Dionissi Aliprantis. This version is from June 2022.

Definition 2: A one-to-one (1-1) correspondence between two sets $\mathcal{A}$ and $\mathcal{B}$ is any rule (function) in which each element in $\mathcal{A}$ is matched with exactly one element in $\mathcal{B}$ and every element in $\mathcal{B}$ is matched with exactly one element of $\mathcal{A}$.

Examples: Students and desks in the room? Fingers on each hand? It is a real simple idea. Name 4 boys and 4 girls from Site 1. List them:

Name 1	Name 1
Name 2	Name 2
Name 3	Name 3
Name 4	Name 4

You can define a rule matching elements, but who is happy with this match up?

1.2 Day 2

Make sure name cards are out, show me if I look at you. All quiet, focus up front, all eyes on me. Wait for silence. Review:

-Cantor (1845-1919) - Died in what type of institution? What did Cantor invent?

TRANSFINITE SET THEORY

What does "transfinite" mean? Q: What did Cantor do? A: He held infinity in the palm of his hand.

Q: Who was the English mystic we spoke about? William Blake (1803) The Auguries (Omen) of Innocence

Redefine a set. Have someone recite the definition of set. "Time to pick up our pencils."

Everything above took about 15 minutes, spent time reviewing.

Definition: Two sets $\mathcal{A}$ and $\mathcal{B}$ are equivalent, or have the same size, if there exists a one-to-one correspondence (MATCH) between their elements.

After writing down, repeat: "Now read with my finger."

Definition: The "size" of a set $\mathcal{A}$ is called its cardinality. Denoted by CARD($\mathcal{A}$).

I need four volunteers to be in the set

$$S = \{\text{Initial1}, \text{Initial2}, ..., \text{Initial4}\}.$$

Tell me each your preferred topping for pizza:

$$T = \{P[\text{Pepperoni}], S[\text{Sausage}], ..., C[\text{cheese}]\}$$

Which set is larger? We can do the obvious thing and match each person with their preferred topping. So there exists a 1-1 correspondence, ie, CARD(S) = CARD(T) = 4.

Here - chance to discuss history and the abstraction from particular sets to the number 4. Sounds simple, but we need to understand it deeply. $\therefore S \equiv T$ "they are equivalent."

So far this is pretty easy stuff...

Now, please, I need all females in the class to count all the males in the class. And I need all males to count all females. Let's figure out:

$$F = \{\text{All females}\} \quad \text{CARD}(F) = 24$$
$$M = \{\text{All males}\} \quad \text{CARD}(M) = 27$$

$\therefore F \not\equiv M$ Let's read the above lines out loud.

What if you didn't know how to count? The Hotentots - possibly only abstracted to 3 (above that only "many"). Let them get to "Just match them up."

INFINITE SETS

Let's get a simple one:

$$\mathbb{N} = \text{the set of natural numbers} = \{1, 2, 3, \dots\}$$

If you say the cardinality of a set is infinity? Are you holding infinity in your hand? No.

Cantor said CARD($\mathbb{N}$) = $\aleph_0$ ("Aleph nought") is a number. Which is the first transfinite number. In your hand? Yes.

ESOTERIC SOCIETY

small/exclusive group, known only to a few

If $\aleph_0$ is the 1st transfinite number, what would be an interesting question?

-What is the 2nd? But Cantor had no idea - Is there another?

Homework:

For tomorrow, $E = \{2, 4, 6, 8, 10, 12, \ldots\}$

Show me with hands, which has larger cardinality, $\mathbb{N}$ or E? Is CARD(E) a 2nd transfinite number? How many more $\mathbb{N}$ are there than E? Twice as many? Question for tomorrow:

$$\therefore \text{CARD(E)} = \frac{1}{2}\aleph_0?$$

What would happen if there were the same size?

QUICK RECAP to close.

1.3 Day 3

"Good afternoon team."

"Good afternoon Mr. Boehm."

INFINITE SETS

Remind the class about the two sets:

1. $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
2. $\text{E} = \{2, 4, 6, 8, 10, \ldots\}$

Q: When do the natural numbers end?
A: It is an infinite set.

Q: Which set is larger? Let's have a discussion.

[Reminder: Look at your colleagues, all of them, and not just me, when addressing the class.]

$\text{CARD}(\mathbb{N}) = \aleph_0$

Q: $\aleph_0$ is what?

A: The first transfinite number.

Q: So what is an interesting question?

A: Does there exist a 2nd transfinite number?

[In response to walking around and seeing a kid doing homework for another class in the back - "I expect better."]

Any conjectures? $\text{CARD(E)} = \frac{1}{2}\aleph_0$? How do we decide when sets have the same size? Let's look at one last finite set.

"We've got to understand simple things in a profound way."

Can someone count the people in the class?

$$P = \{\text{People in the Room}\}$$
$$D = \{\text{Desks in the Room}\}$$

CARD(P)=50. Note: You got this by counting.

CARD(D)=52. $\therefore P \not\equiv D$.

Simple Q: Suppose you didn't know how to or simply couldn't count? How could you determine whether they are of the same size?

In this class we get to investigate simple-looking ideas that turn out to not be so simple. Let's list them out:

$$\begin{array}{rl} \underline{E} & \underline{\mathbb{N}} \\ 2 & 1 \\ 4 & 2 \\ 6 & 3 \\ 8 & 4 \\ 10 & 5 \\ 12 & 6 \\ 14 & 7 \\ 16 & 8 \end{array} \tag{1}$$

I can match 2 with 2, etc. Obviously, they are not matching up.

Q: From this, can I conclude that CARD(E) $\neq$ CARD($\mathbb{N}$)? (ie, E $\not\equiv \mathbb{N}$)?

Q from the audience (Jamaul): Can 2 be matched with and 4 with 2, etc.?

DISCUSSION follows. Jamaul makes the point: "They're both infinite, so you can't run out of numbers."

If Jamaul is correct, then this will seem to mean that every infinite set has the same cardinality!

In this case, how many transfinite numbers will there be? Only 1?

Timeout - praise for thinking.

Someone else: "But you are leaving some numbers out! I think there exist different levels of infinity."

Q: Is 2 a 1-1 correspondence?

A: No.

Q: Does that [the face that the correspondence is not 1-1] prove that there does not exist a 1-1 correspondence? Can we conclude there does not exist a 1-1 correspondence from the above question? [Spent 5-10 minutes re-phrasing this question in various forms, using the word MATCH, etc.]

"Pick up your pencils." I want to claim that ℕ and E have the same cardinality. Write as I go... ["Gang it is too noisy."] [Now matching them straight across.]

$$\begin{array}{cc} \underline{E} & \underline{\mathbb{N}} \\ 2 & 1 \\ 4 & 2 \\ 6 & 3 \\ 8 & 4 \\ 10 & 5 \\ 12 & 6 \\ 14 & 7 \\ 16 & 8 \end{array} \tag{2}$$

If I keep going forever, there exists a MATCH!! So $\mathbb{N} \equiv E$ (ie, CARD(ℕ) = CARD(E) = $\aleph_0$)

Welcome to Weirdness!!

Q from the audience: Would it be the same for odd numbers?

[After writing on the board, read this out loud following your finger to end.]

WEIRD FACT #1: CARD(ℕ) = CARD(E)

Q: Why is this WEIRD/counterintuitive (ie, something that doesn't seem right at first)

A: Because there are "twice as many" natural numbers than even numbers, but they have the same size.

Q: What would be an interesting question?

A: Are all infinite sets the same size?

HOMEWORK: Can you think of an infinite set with different size than ℕ?

1.4 Day 4

1.4.1 Review

What is the definition of a set?

Cantor said a set is a many that can?

What did Cantor do? [hold infinity in the palm of his hand.]

What did he invent?

What does transfinite mean?

What is the size of $\mathbb{N}$?

$\aleph_0$ is the first what?

Will there be another transfinite number?

The size of a set is called its? [Cardinality]

How do we know when two sets have the same cardinality?

How do we say 1-1 in colloquial language? [MATCH]

1.4.2 Class

Why is that a weird fact? "Pick up your pencils."

Definition: A set $\mathcal{A}$ is COUNTABLE (ie, denumerable) if either:

1. the set is finite
2. $\text{CARD}(\mathcal{A}) = \aleph_0$

What else might work for an infinite set? [$\mathcal{A} \equiv \mathbb{N}$]

What would it mean for a set to not be countable?

Examples: 1) Odds $= \{1, 3, 5, 7, 9, \ldots\}$

CARD(Odds) = ? NOTE! This question could not even be asked until Cantor! [$\aleph_0$]

Proof: [What do we have to do to prove this?] Simple idea:

$$\begin{array}{cc} \underline{\mathbb{N}} & \underline{Odds} \\ 1 & 1 \\ 2 & 3 \\ 3 & 5 \\ 4 & 7 \\ 5 & 9 \\ \vdots & \vdots \\ 100 & ?[199] \\ \vdots & \vdots \\ N & ?[2N-1] \end{array} \tag{3}$$

(4)

1) Primes $= \{2, 3, 5, 7, 11, 13, \dots\}$

CARD(Primes) $= \aleph_0$

Proof: Another two columns...

What does N go to? This is a very deep problem! $\therefore$ CARD(Primes) = CARD(Evens) = CARD(Odds) = CARD($\mathbb{N}$) = $\aleph_0$

"Would you pick up your pencils and write:" Does there exist another transfinite number besides $\aleph_0$? The cool thing is (!): Cantor had no idea where this would lead! Like doing research today.... Where would he look? Consider the integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. [They note - there is not beginning and there is no end. "With other sets you have got somewhere to start."] So maybe a proof by contradiction? Is this set the same size as $\mathbb{N}$? What do you think? Let's vote. I'll give you a hint and leave this open for the weekend.

HINT: THE SOUPY SHUFFLE

Question: How can I write a proof? The hint is to think of the Soupy Shuffle.

1.5 Day 5

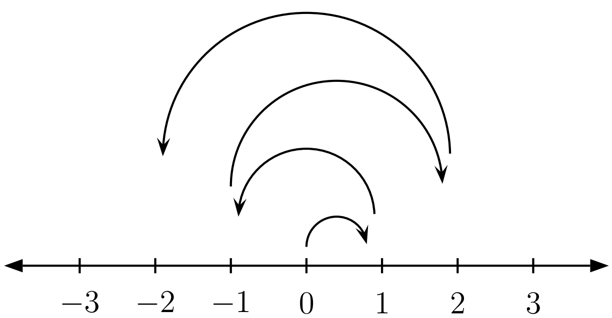
1.5.1 Review

Let's take 30 seconds to go through yesterday's notes. Len walks through the class while the students look at their notes. What is the interesting question we want to attack today?

Is the set of integers countable?

If it were countable, what would we have to show? [$\exists$ a 1-1 correspondence with $\mathbb{N}$]

Why might $\mathbb{Z}$ not be countable? [Because unlike the other sets it has no beginning and no end.] Who is the famous person who is going to help us figure this out? Soupy! The Soupy Shuffle is the big hint. Len let Jamaul come up and teach the class, as he had figured it out earlier.



$\mathbb{N}$	$\mathbb{Z}$
1	0
2	1
3	-1
4	2
5	-2
$\vdots$	$\vdots$

"You were right, it is confusing at first." Who can tell us what 100 would get matched with? [50 - evens go to positives] What about odds? This may seem like a simple thing, but you are now in an ESOTERIC SOCIETY. Not a lot of people understand this. "Pick up your pencils, let's get this down in our notes:"

WEIRD FACT # 2

$\text{CARD}(\mathbb{Z}) = \text{CARD}(\mathbb{N}) = \aleph_0$

Can someone explain why this is a WEIRD fact? [Because the set of integers is “twice as large” as the naturals but they have the same size.] This should help you to appreciate Cantor. He had no idea what would happen. One thing though - if every infinite set had the same cardinality, would this class be interesting? Before Cantor you couldn’t even ask this question! (ie, Is there a non-countable set?)

Now let’s consider another set. Any suggestions? Steer them to the rationals $\mathbb{Q}$. What is the definition of a rational?

$$\mathbb{Q} = \{ \text{ All fractions } \}$$

Examples:

$$\frac{1}{2} = 0.5\overline{0}$$
$$\frac{3}{4} = 0.75\overline{0}$$
$$\frac{2}{7} = ?$$

Think about Cantor. He’s looking at the set and he realizes $\mathbb{Q}$ has a property the other sets don’t. Let’s appreciate this. How many natural numbers are there between 3 and 8? [4] More generally, how many numbers are there between and 2 natural numbers? [a finite number] What property do the rationals have that the naturals do not?

“Let’s get this in today’s notes:” Note: The rationals form a DENSE set. Can someone define DENSE based on the answer to the earlier question? “All eyes on Jamaul.” Between any two rational numbers there exist an infinite number of rational numbers. Let’s just look at some numbers on the unit interval. How many numbers are there between $\frac{1}{4}$ and $\frac{1}{2}$?

Who thinks CARD($\mathbb{Q}$) ¿ CARD($\mathbb{N}$)? Discuss

1.6 Day 6

Mr. Boehm lets the students know that at the university, good students are ready and already thinking when the class is scheduled to start. They have already spent 3-4 minutes going over yesterday’s notes and thinking about what might come up today.

1.6.1 Review

Is the set of students finite?

Can we count them? What about the desks in the room?

If a set is finite and countable, we can do what?

"Nice support."

For an infinite countable set, there exists a what? [1-1 correspondence with $\mathbb{N}$]

In that case the cardinality of the set is? [$\aleph_0$]

Are the evens countable? Why is that a weird result? [they have "twice as many" elements]

What about the odds? $\mathbb{Z}$? Are they countable? Why is that a weird result? [No beginning *and* no end.]

Since every single infinite set we've looked at so far has cardinality equal to $\aleph_0$, what is an interesting question? [Does there exist a set that is not countable (ie, whose cardinality is not equal to $\aleph_0$)?]

What was the set we were considering from yesterday?

1.6.2 Class

"Pick up your pencils:"

$$\text{Is } \mathbb{Q} = \{ \text{ All fractions } \} \ \_\_\_\_? \ \text{[countable]}$$

Why would we suspect that we can't count $\mathbb{Q}$? Anyone remember the property of the rationals that the other numbers don't have? [Density: There is an infinite number of rational numbers between any two rationals.]

Can someone give me a decimal close to 0? Class gives:

0.75

0.01 [can we get closer?]

0.00001 [can we get closer? How many 0's can we add? what if there were 83 - the greatest number in all of WSU football history. Are these all still in $\mathbb{Q}$? How many rationals live between 0 and 0.00...(83 zeros)...0001?]

Now, if $\mathbb{Q}$ were not countable, there would have to exist what? [A second transfinite number.]

After a student provided that answer, Herman asked: "What did she say?" Mr. Boehm responds by putting Herman's name on the board, and asks - "Why did I feel compelled to put Herman's name on the board?" ["Let's use people's names rather than impersonal pronouns [like he/she]."]

On the other hand, if the rationals are countable, what must exist? [a 1-1 correspondence with the $\mathbb{N}$]

If they were not countable, what type of proof would we try to write? [Proof by contradiction.]

Mr. Boehm goes through and asks the class whether they think the rationals are countable. There is a long DISCUSSION. "That is a great question, can someone repeat it?" "If there exists a second transfinite number, it can't be equal to $\aleph_0$, can it? But how do we get to 1? And if we can't get to 1, how can we get to any of the other natural numbers?"

OK, give me two minutes (in a rush). Let's write out one part of the rational numbers this way - Let's start with all fractions with 1 in the denominator [Let's start with just the first 6]:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \ldots$$

This is one infinity. Can we count this infinity? Now suppose I add all the guys who have 2 in the denominator? Can we shuffle? Can we match up 2 infinities? Can you match up all

$$\begin{array}{ccccccccccccc}
\frac{1}{1}, & & \frac{2}{1}, & & \frac{3}{1}, & & \frac{4}{1}, & & \frac{5}{1}, & & \frac{6}{1}, & & \ldots \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
\frac{1}{2}, & & \frac{2}{2}, & & \frac{3}{2}, & & \frac{4}{2}, & & \frac{5}{2}, & & \frac{6}{2}, & & \ldots
\end{array}$$

possible fractions?

1.7 Day 7

1.7.1 Review

Recall $\mathbb{Q} = \{\text{All fractions}\}$. Is CARD($\mathbb{Q} = \aleph_0$? Is $\mathbb{Q} \equiv \mathbb{N}$? Remember, this represents an infinite number of infinities (not just a doubling like evens relative to naturals, or negatives relative to positives for integers).

1.7.2 Class

Let's ask: Can we count 3 infinities?

$$\begin{array}{ccccccc}
\frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \frac{4}{3}, & \frac{5}{3}, & \frac{6}{3}, & \ldots \\
\frac{1}{2}, & \frac{2}{2}, & \frac{3}{2}, & \frac{4}{2}, & \frac{5}{2}, & \frac{6}{2}, & \ldots \\
\frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \frac{4}{1}, & \frac{5}{1}, & \frac{6}{1}, & \ldots
\end{array}$$

That's what Cantor did! With integers we counted 2 infinites. Can we count 3? Raven suggests

$$\begin{array}{cc} \underline{\mathbb{N}} & \underline{\mathbb{Q}} \\ 1 & \frac{1}{1} \\ 2 & \frac{1}{2} \\ 3 & \frac{1}{3} \\ 4 & \frac{2}{1} \\ 5 & \frac{2}{3} \\ 6 & \frac{3}{1} \end{array} \tag{5}$$

or Why did Jozy skip $\frac{2}{2}$? [Equivalent fractions to $\frac{1}{1}$] Jamaul: But I had an idea. It seems like

$$\begin{array}{cccccccc} \frac{1}{3}, & \rightarrow\downarrow & \frac{2}{3}, & \frac{3}{3}, & \frac{4}{3}, & \frac{5}{3}, & \frac{6}{3}, & \cdots \\ \uparrow & \downarrow & \uparrow & & & & & \\ \frac{1}{2}, & & \frac{2}{2}, & \frac{3}{2}, & \frac{4}{2}, & \frac{5}{2}, & \frac{6}{2}, & \cdots \\ \uparrow & \downarrow & \uparrow & & & & & \\ \frac{1}{1}, & \rightarrow & \frac{2}{1}, & \frac{3}{1}, & \frac{4}{1}, & \frac{5}{1}, & \frac{6}{1}, & \cdots \end{array}$$

you can count any finite number of infinities. If it is finite, you always have a place to stop at. Would someone like to address the class and repeat what Jamaul just said? Herman: So basically, the problem is that you can't count an infinite set of infinities? Because in that case once you go up, you'll never be able to come back down!

Mr. Boehm: So who thinks they are not countable?!? [Very excited, as if a big truth/great insight has been discovered.] WRONG!! Stop! I need everyone... What is the cardinality of the first row? The second? The third? How many rows are there? [An $\aleph_0$ number of rows, each of cardinality equal to $\aleph_0$] Now Len puts up an overhead and hands it out. Raven comes to the board and just fills it out like it is not big deal. [The handout is nice because it lends itself to a nice spiral pattern, with 0 in the middle with the positive and negative rationals in the northeast and southwest corners, respectively.] Great proof Raven!!!!

So who thinks $\aleph_0$ is the only transfinite number? What is the next set we might want to think about when trying to answer this question? Next we'll think about the reals, which can be split into which two disjoint sets? [$\mathbb{R}$ = Rationals $\bigcup$ Irrationals] Which can also be thought of as all repeating decimals [and/or terminating] and all non-repeating infinite decimals.

Some examples of repeating decimals: $\frac{3}{4} = 0.75\overline{0}$, $\frac{1}{3} = 0.\overline{3}$

Example of a non-repeating decimal: 0.101001000100001... This one has a pattern, but does the pattern repeat? So please, show me what you think: Is $\mathbb{R}$ countable? By Monday: If they are countable, can you show me a 1-1 correspondence with $\mathbb{N}$? If not, what kind of proof are we going to have to write? [Proof by contradiction.]

1.8 Day 8

1.8.1 Review

Can you please go over your notes from last class at least one time before we start?

"Good afternoon team."

"Good afternoon Mr. Boehm."

Who is the godfather of soul? [James Brown] Who is the father of transfinite set theory? [Georg Cantor]
He held infinity in the palm of his? [hand]
What was the first set we started thinking about? [$\mathbb{N}$]
What was the first transfinite number? [$\aleph_0$] Then we looked at? [$\mathbb{Z}$] Then? [$\mathbb{Q}$] And their cardinality was always? [$\aleph_0$]

1.8.2 Class

So what is the BIG QUESTION facing Cantor? Remember, he had no idea what the answer would be.

BIG ONTOLOGICAL QUESTION (ONTOS $\approx$ "exists", LOGICA $\approx$ "the study of"):

DOES THERE EXIST ANOTHER TRANSFINITE NUMBER BESIDES $\aleph_0$? [Read together, stress that it is so we can focus. "Thanks, I appreciate you all appreciate why it is important we read."]

If such a number exists, then there would have to be a set that what? [You couldn't count] Now, let's prepare for weirdness. Q: Is the set of reals [countable]? Let's break that down. 1st Question: What is a real number?

$$[\mathbb{R} = \{\text{all decimal numbers}\}]$$

We can break this into two disjoint sets: $\mathbb{R} = \{\text{Rationals}\} \bigcup \{\text{Irrationals}\}$
RATIONALS: Their decimal representation has what property? [Repeating (or terminating) decimals]

Examples: $\frac{2}{3} = 0.\overline{6}$

$\frac{5}{4} = 1.25\overline{0}$

IRRATIONALS: Their decimal representation has what property? [Non-repeating infinite decimals]

Example: 0.123456789[10][11][12][13]... Who sees the pattern?

I'm going to show another decimal with a pattern but that doesn't repeat:

0.1010010001... Who sees the pattern?

What's the most famous? [$\pi = 3.14159...$ maybe discuss the method of exhaustion]

So let's return to our question: Can we count the reals? Someone give me a reason we could.

Someone guesses: They don't seem to be as dense as the rationals. Someone else: But does it matter? Won't they just get caught up in Raven's spiral from last class? Someone else: But rationals were an infinite number of infinities, so we should be able to. Len points out: But if you're pulling π out of a jar, you'd be pulling it out forever... To which someone asks: So can you truly even move on to the next number? Raven asks: But where do you start at listing them if they are never repeating? Len: Why don't we just start counting them?

Get this down in today's notes: The set of reals is called the CONTINUUM. That is because once you get the irrationals in there, there is nothing else you can squeeze in, no holes, you've got everything. "Let's pick up our pencils and ask this question:" Just focusing on the unit interval of all reals in $(0, 1)$, is the set $S = \{(0, 1)\}$ countable?

Let's suppose we could count them. Let's start by blindly picking them out (answers given by class):

$$r_1 = [0.1\overline{0}]$$
$$r_2 = [0.101001...]$$
$$r_3 = [0.1234...]$$

If the set is not countable, then that implies there is some guy in there I will never get, ie, they can run forever, and ever, and ever. Does that strike you as so WEIRD it is impossible? Well the answer is NO!!!

DISCUSSION: Class, there are three things I want to accomplish with this class.

1. Celebrate your abilities as a thinker

2. I wanted everyone to understand Cantor's first big theorem

3. " " Cantor's second big theorem

How can we show this runaway number exists? Question of existence - ontology. We have to prove that what does not exist? Raven: A 1-1 correspondence with the $\mathbb{N}$. Question: How to prove something does not exist? Elena: Proof by contradiction. Len hands out the first take home test (for fun, not to be collected.).

1.9 Day 9

1.9.1 Review

Can someone spell ONTOLOGICAL? (existing, study of)
What was Cantor's big step? [$\aleph_0$]
What is the size of $\mathbb{N}$? [Not everyone! I need everyone!]
What about the set of all evens? Primes? $\mathbb{Z}$? $\mathbb{Q}$?
So every single infinite set we have examined has cardinality what? [$\aleph_0$]
What was the subset of the continuum we want to look at? [reals in (0,1)]

1.9.2 Class

Pick up your pencils, here we go.

Theorem: The set of all reals in (0,1) is? [countable]

Jamaul: But?!?

Len: Who thinks this was a senior moment? And who thinks this was a subtle pedagogical technique to make sure you all are sharp?

Theorem: The set of all reals in (0,1) is? [UNCOUNTABLE]

That implies that what does not exist? [A 1-1 correspondence with the natural numbers]

Discussion of three things want to do with this course (celebrate as thinkers, understand Cantor's first and second theorems). By the end you all will be members of an ESOTERIC society [def: only known to a few people.]

But we need to know 3 things:

I Simple Simon Says (SSS)

Take the unit interval and write a real number from it:

$r_1 = 0.XXXX\_XXX$ [8]

Just write down the first 8 digits. I just want to know what is in the 5th digit:

$r_2 = 0.XXXX\_XXX$ [6]

Can someone make a statement about the numbers? Could they possibly be the same?

$\therefore r_1 \neq r_2$

II Proof by Contradiction (PBC)

1st step: Assume the opposite of your statement is true

2nd step: You have to derive/arrive at a contradiction (I will use the symbol X, also used is $\rightarrow\leftarrow$)

3rd step: Draw a conclusion ($\therefore$ the opposite of your statement is false)

4th step: $\therefore$ your statement is true

Example (Easy): The set of natural number is? [infinite] Go through the steps

III Dodgeball (From "The Heart of Mathematics", have them play)

Player 1 fills in row 1. Player 2 fills in row/column 1.

Proceed until Player 1 has filled in all six rows and Player 2 has filled in a row. Player 2 wins if their row does not appear in any of Player 1's rows. Player 1 wins if it does appear there.

1.10 Day 10

1.10.1 Review

[visitor's present]

What was one of the greatest poems in the English language? [The Auguries of Innocence]
Who was its author? [William Blake]
Who was the first person to hold infinity in their palm of their hand? [Georg Cantor]
He said that CARD($\mathbb{N}$) = ? [$\aleph_0$]
What is $\aleph_0$? [The first transfinite number]
Transfinite means? [beyond finite]
What was the second infinite set we looked at? [Evens]
Why is it weird that CARD($\mathbb{N}$)=CARD(Evens)? [On first glance, it looks like there are twice as many elements]
What was the next set we studied? [$\mathbb{Z}$]
Why was that weird? [There were both negatives and positives]
Can you all turn, wave your hands, and smile at the student who proved $\mathbb{Z}$ are countable? [Everyone turns and waves at Jamaul]
Then what set of numbers did we look at? [$\mathbb{Q}$]
What was weird about them? What different property do they have? [Density]
[Writes "ONTOLOGY" on the board]: Can someone read this? Tell me what it means? [The study of existence]

What is the BIG ONTOLOGICAL QUESTION we have been thinking about? [Is $\aleph_0$ the only transfinite number?] Remember, Cantor had no idea...
What was the next set we looked at? [$\mathbb{R}$]
In particular, what part did we focus on? [Unit interval - $(0, 1)$]

1.10.2 Class

Is the set of reals in $(0, 1)$ countable? What does it mean to say a set is countable? [$\exists$ a 1-1 correspondence with $\mathbb{N}$] ie, is CARD(0,1)=$\aleph_0$ Answer: NO.

Can someone explain this weirdness?!? If I look at all reals in (0,1), what does it mean to say that they are not countable? [$\nexists$ a 1-1 correspondence with $\mathbb{N}$] Let's pick a couple reals:

$$r_1 = []$$
$$r_2 = []$$
$$r_3 = []$$
$$\vdots \quad \vdots$$

If it is countable, can we count forever and ever and ever? So if it was uncountable, that implies there exists some number on the unit interval so that? [It never gets picked, even when you go on forever and ever and ever... D's note - can say it is like a glitch in the Matrix - on the run forever and ever and ever...]

Julius: But there is no first number!

Len: Did that happen with any other sets?

To prove something doesn't exist, we are going to use Proof By Contradiction (PBC). Our proof needs three things:

1. SSS
2. PBC
3. Dodgeball

Turns out, Julius and Jamaul were playing dodgeball the night before and figured out Player 2's winning strategy. They calmly come to the front of the room and explain it. Then Len re-explained. Then Raven quietly says - "Oh, I get it." Then she explains to the class the basic outline of Cantor's diagonalization proof. [Incredibly cool discussion to sit and watch....]

1.11 Day 11

1.11.1 Review

"Good afternoon team."

"Good afternoon Mr. Boehm."

Team 1? "I feel good."

Team 2? "Like I knew that I would."

A couple comments giving positive reinforcement (PR) for the cool way the students interacted with the visitors.

Pick up your pencils. We want to prove CANTOR's 1st GREAT THEOREM.

Theorem: The set S of real numbers in (0,1) is [uncountable.]

Reminder: Nobody could even ask this question before Cantor, let alone come up with the answer. We are going to do a proof by [contradiction.] So we need to prove there does not exist a 1-1 correspondence between the natural numbers and (0,1).

Cantor's Diagonalization Proof (PBC):

Circulating through the room, asks: What's the first thing to do in a PBC? You can look in your notes.

1st step: [Assume the opposite of your statement is true,] so assume S is [countable]

2nd step: [You have to derive/arrive at a contradiction,] so under this assumption there exists a 1-1 correspondence between [$\mathbb{N}$ and S]. Will you ever miss a number?

Give me the first 6 digits of a decimal you pick. Now give me a second number (just the first 6 digits):

$$
\begin{array}{l}
\underline{\mathbb{N}(0,1)} \\
1 \rightarrow [0.135791...] \\
2 \rightarrow [0.364592...] \\
3 \rightarrow [0.247813...] \\
4 \rightarrow [0.100196...] \\
5 \rightarrow [0.123479...] \\
6 \rightarrow \\
\vdots
\end{array}
$$

If it is countable, then there does not exist a guy we will miss. We'll get everyone. Everybody OK? Show me signals. To get a contradiction, that means we need to find a missing real, let's call it M. Let's suppose M only uses the digits 1 and 2. The Diagonalization Proof goes as follows: So we can focus, can someone come up to the board and for the above expression, just circle the digits on the diagonal? Is there anyone who thinks they know the next step? [Beautiful! Jamaul explains the proof in terms of the game dodgeball. Then Raven embelishes and adds some nuance.]

Let's pause and reflect on how dodgeball was played. Raven explains it! M=0.21121... Len goes over it slowly: "Now, could it possibly be the 2nd number?" Remember, it (ie, M) goes on forever and ever. For r_3, could it be 1? 2? r_4: Now I don't have a choice. As we go on, we are creating this infinite decimal. If we continue this process, will M ever be in this list? Jamaul: Without any restrictions implies there exist an infinite number of Ms not in the list. William proceeds to explain how M will never be matched. $\rightarrow\leftarrow$ We just created a what? [Contradiction, go over]

3rd step: $\therefore$ "S is countable" is False

4th step: $\therefore$ "S is uncountable." QED

Question from Julius: But should there be a 5th step? That there exists a 2nd transfinite number? Discussion starts and Len gets excited and gives PR (positive reinforcement) for asking a good question. Discussion is resolved with - Instead of a 5th step, we can say $\therefore$ CARD(0,1) $\neq$ [$\aleph_0$]! It is a higher infinity!

Since CARD(0,1) = c, Jurrell says it must be the 2nd transfinite number. Raven: So will CARD($\mathbb{R}$) > CARD(0,1)?

Let's get the question for Monday down, which is the question you are going to ask next if you are Cantor. This is a BIG ONTOLOGICAL QUESTION (Jordan states it): Does there exist a 3rd transfinite number? Any thoughts? Camyrea: Does there exist an ordering of these numbers? Like $\aleph_0 < c <?$, discussion follows with Jamaul saying that it seems like there is more of a classification, just $\aleph_0$ for countable numbers and c for uncountable. What would be the next set to look at? [(0,2)] And the question then? [CARD(0,2) $\overset{?}{>}$ c or CARD(0,2) $\overset{?}{=}$ c?]

1.12 Day 12

Len waits for the class to get quiet.

1.12.1 Review

"Good afternoon team."

"Good afternoon Mr. Boehm."

Team 1? "I feel good."

Team 2? "Like I knew that I would."

What was the first transfinite number? [$\aleph_0$]

WEIRD FACT # 4: CARD(0,1) = $\aleph_0$

Everyone who thinks this is true, read with my finger. Now everyone who thinks this is false, read with my finger.

With signals, is the equation on the board true or false? What did we prove last week? [Elena says CARD(0,1) $\neq \aleph_0$]

What does that imply? Why is that WEIRD?!? [The WEIRD fact is that the real numbers between 0 and 1 are uncountable!]

Can a good listener repeat that? So CARD(0,1) = [c], and so [c is the 2nd transfinite number.]

Why is that WEIRD? (ie, to say that the reals are uncountable) [Because uncountability is just plain WEIRD!]

Does anyone remember the next BIG ONTOLOGICAL QUESTION Cantor was going to ask? [Jordan: Does there exist a 3rd transfinite number?]

1.12.2 Class

Pick up your pencils and write:

"THE JAMAULIAN CONJECTURE"

[A set is either countable or uncountable.]

In other words, there only exist two transfinite numbers, or two levels of infinity, $\aleph_0$ and c

Can someone support Jamaul's thinking? Julius: Anything countable has cardinality equal to $\aleph_0$ and anything uncountable has cardinality equal to c.

Who thinks there exists a 3rd transfinite number? Tyrique: Well I think there's nothing in between. (Len, Dr. Steve, Dr. D all very happy excited by this!) Can a good listener please repeat this?

Now, can anyone give me another set we might want to think about? [(0,2)] Next, let's consider the reals between 0 and 2: CARD(0,2) = ? Jamaul: probably 2c, but we tried with integers and saw that it wasn't equal to $2\aleph_0$, so it is probably the same.

To refocus, could someone repeat that in their own words? Raven: There is a parallel between

-ℝ and (0,1) and

-ℚ and ℕ

in the sense that there is an infinite number of infinities.

Len: But we are dealing with uncountable sets! Who thinks CARD(0,2) = 2c? CARD(0,2) $\neq$ 2c? What do we have to do to prove one or the other?

Elena: If c is uncountable, how can you find a 1-1 correspondence? Mikayla: It would have to be proof by contradiction.

Len: Let's go back to the start, what does a 1-1 correspondence imply? [there exists a match] Starts talking about back in the day, how this relates to the "invention" of numbers.

Shanese: I'm confused about what we're even trying to prove!?!

Kera: But how can we even get past (0,1) because we jus had a PBC?

Raven: I think CARD(0,2)=c because an infinite number of numbers get left out, just like for (0,1)

Camyrea: Does that imply CARD(ℝ) $\neq$ c?

Tyrique: Why can't we just say it is uncountable? (Lots of agreement)

Len: Let's discuss for a minute and then refocus on Jurrell.

Jurrell: We all agree (0,2) is uncountable. How can something be more uncountable than something else?

William: I agree, how can something have cardinality greater than c?

Mikaela: I respectfully disagree. I think there are more infinities between (0,2) and (0,1)

Len: So the question for tomorrow is: If we're going to prove this, what do have to find/show?

QUESTION (again, an ONTOLOGICAL QUESTION): Does there exist a 1-1 correspondence between [(0,1) and (0,2)]?

Last question: If this is false, how would you show it?

1.13 Day 13

Len waits for the class to get quiet.

1.13.1 Review

"Good afternoon team."

"Good afternoon Mr. Boehm."

Team 1? "I feel good."

Team 2? “Like I knew that I would.”

On the count of 3, get prepared for weirdness. Circulates, I need names cards, etc.

Can someone with good handwriting write on the board

THE JAMAULIAN CONJECTURE

[Raven: There only exists two transfinite numbers (ie, two levels of infinity), $\aleph_0$ and c]

Len: Nice! You guys are on the money, I'm proud and inspired by how well you are doing with this stuff. Why did he think that might be true?

Daemon: Because you just can't count something if it is uncountable.

Len: What famous proof did we employ when we showed there exists an uncountable set?

Jurrell: Cantor's diagonalization proof

Can anyone remember Tyrique's statement based on this belief in Jamaul? The JAMAULIAN CONJECTURE $\implies$

THE TYRIQUIAN HYPOTHESIS

There does not exist a transfinite number between [$\aleph_0$ and c]

So what is the set we want to look at today? [(0,2)] So the Question of the Day is? [CARD(0,2)=?] What does your intuition tell you?

Jurrell: c

Can someone give a reason why it might not be c?

Because CARD(0,1)=c.

Julius: It includes (0,1)

Tyrique: Something included in it is uncountable

Alaina: But if it is uncountable how can we make a 1-1 correspondence?

Tyrique: But c just means uncountable.

Alaina: But they're not equal!

Camyrea agrees: But they're uncountable for the same reason, so I think there does exist a 1-1 correspondence.

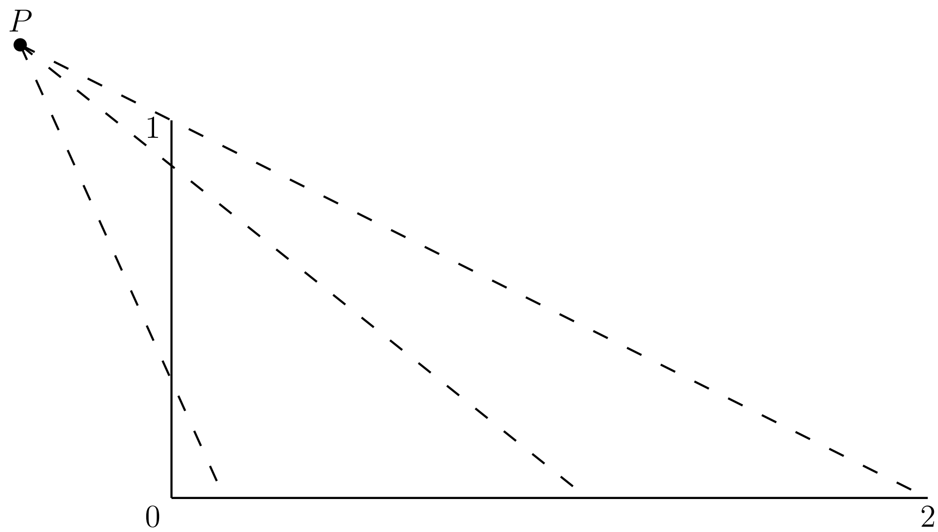
Jamaul: But it's like undefined! Like $\frac{2}{0} = \frac{3}{0}$, both are undefined!

Len: Don't get frustrated - it is confusing!

Leslie: But the definition of countable is a 1-1 correspondence <u>with $\mathbb{N}$</u>, so they're uncountable, can't match with $\mathbb{N}$.

Len: But do they have the same size? Let's have a cat's meow – turns out that yes, $CARD(0, 2) = c$. But, what does that mean exists? [a 1-1 correspondence]

Shanise: But how can a 1-1 correspondence exist?



Proof (Geometric):

We want to match up every point in (0,1) with a point in (0,2). Anyone see how to do that?

Take a number in (0,1). I have to match it with a number in (0,2). Do the same thing with a number in (0,2). How to determine the match? Denay answers perfectly - drawing lines - OH!

Camyrea: So we eventually get to every point in (0,1) and (0,2)!

Len: But is there anything magical about (0,2)?

Shanise: I don't get it.

Len: Can someone explain that better than me?

Dr. Steve: Two lines can intersect in only one point.

Ivan: Then why can't it be $\mathbb{N}$?

Raven: ie, Why can't you use the geometric proof to show $CARD(0, 2) = CARD(\mathbb{Q})$?

Len: But we don't know that line connects an element of $\mathbb{Q}$ in (0,1) with an element of $\mathbb{Q}$ in (0,2). Does this proof still work for (0,3)? (0,10)?

Jurrell: Explains where P in the figure comes from.

Len: So how much of the real line does this work for?

Donte: All of it.

Raven: [Notices that for $(-\infty, \infty)$ you can use the circle for a proof.]

Len: Can someone who was listening repeat that?

Camyrea does. Pencils please, one last thing. The question for tomorrow: Based on what Raven just said, the CARD($\mathbb{R}$)=? Proof?

1.14 Day 14

"Good afternoon team."

"Good afternoon Mr. Boehm."

Team 1: "I feel good."

Team 2: "Like I knew that I would."

1.14.1 Class

Positive Reinforcement for Shanise for asking and saying that she felt confused.

WEIRD FACT #5

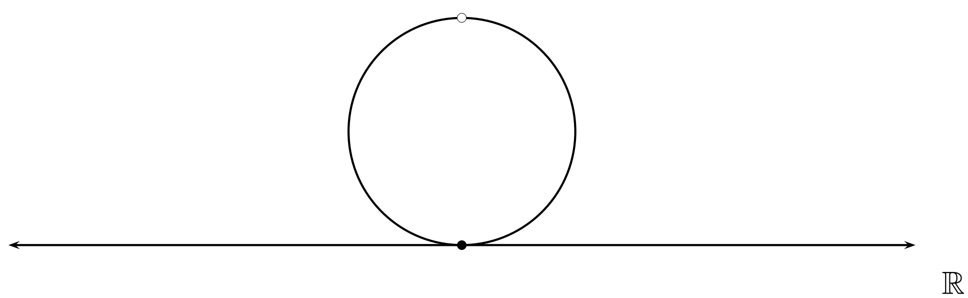
CARD($\mathbb{R}$) = CARD(0,1) = c

c is for cardinality of the CONTINUUM

Geometric Proof: Anyone remember Raven's idea? [Bailey: We can use a circle to do it.]

Len: Consider the half-open interval (0,1) (in picture form) as a string

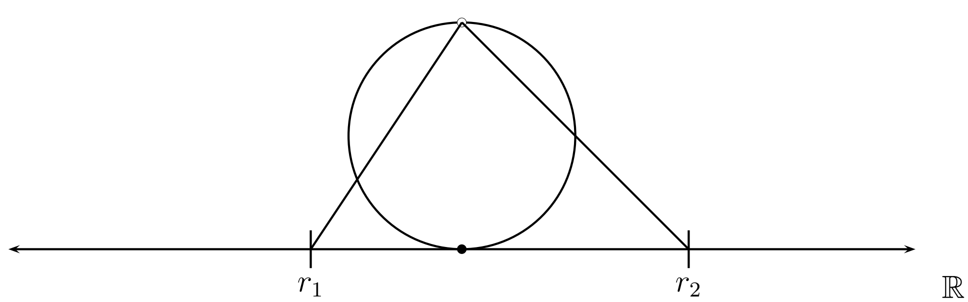
I can make all kinds of shapes, but the length is still 1. Let's sit the string on a number line. What is the circumference of this circle? [1]



Any number in $\mathbb{R}$, you want to match it with a number in (0,1). Any ideas?

Raven: Draw a line from the point to the circle.

Len: Can someone repeat that?



Then if I give you a positive $r_2 \in \mathbb{R}$, what to do? Alright, now get set for real weirdness!

Pick up your pencils and write:

IT EVEN GETS WEIRDER (with picture of Cap'N Weirdo on the side)

Is there anything magical about (0,1)? Could I have used (0,2) or some other interval?

What about (0,$\frac{1}{10}$)? (0,$\frac{1}{100}$)? (0,0.00000083)? But that's the point! I can use any interval (0,A)!

How small can A be? [As small as you want]

So the cardinality of the continuum is the same as the size of the smallest interval imaginable! Let's get poetic [William Blake - start of Auguries of Innocence]

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour

WEIRD FACT #6

"CARD(World) = CARD(grain of sand)"

(ie, $\mathbb{R} \equiv (0, A)$)

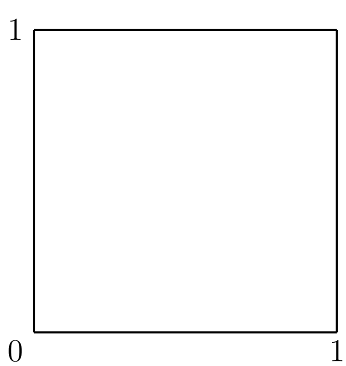
Restatement of "Jamaul's Hypothesis"

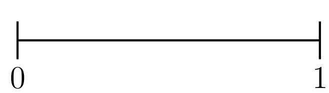
(Buffo the clown completely messes this up. Comes in with a beard and thinks he is supposed to be Cantor/Jamaul's great-grandfather. Len is just shocked that Buffo can't get one (!) line right, and says, "No - I'm supposed to be Cantor!" [I was suppossed to be William Blake and Len was supposed to be Cantor. There's always next year...])

c=cardinality of the CONTINUUM

CANTOR thought: Maybe if he went to higher dimensions he could find another infinity. Cantor believed the 2nd dimension had a higher cardinality than the 1st dimension. How could he prove that?

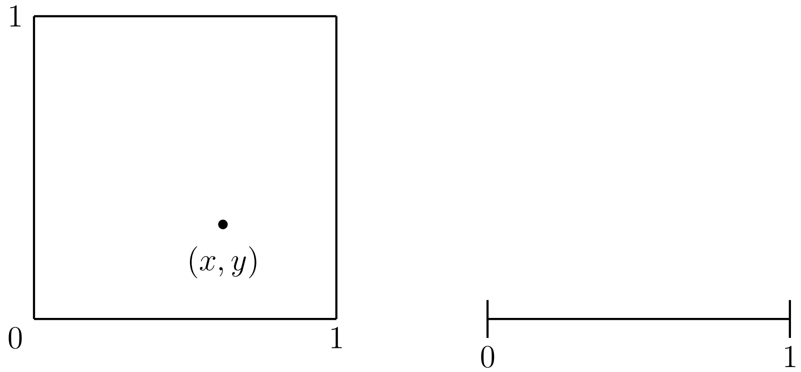
Donte: He would use proof by contradiction (PBC) to prove that there does not exist a 1-1 correspondence.





Cantor went 3 years studying every day trying to figure out a PBC.

After all that time and work he found to his surprise that there does exist a 1-1 correspondence. Once he proved it, he said: "I see it, but I don't believe it."



Who helped Georg? [Soupy!] What did he do? [Shuffle!]

Handout on how to construct a 1-1 correspondence like a zipper to combine x=0.13579... and y=0.2468... to get 0.123456789....

<u>WEIRD FACT #7</u>

$\mathbb{R}^2$= all points in the plane/2-dimensions (explains)

CARD($\mathbb{R}^2$)=CARD($\mathbb{R}$)=c

Camyrea: Does that mean all dimensions have CARD=c?

1.15 Day 15 (not 16 because one day was entirely review)

"Good afternoon team."

"Good afternoon Mr. Boehm."

Team 1: "I feel good."

Team 2: "Like I knew that I would."

1.15.1 Review

What are the 2 transfinite numbers we know about? [$\aleph_0$ and c]

What was the Jamaulian Conjecture?

The Tyrequian Hypothesis?

The 1st four lines of the Augyries of Innocence?

<u>WEIRDNESS</u>

"CARD(World) = CARD(A Grain of Sand)"

That is a poetic statement, but what does that mean mathematically?

Donte: CARD($\mathbb{R}$) = CARD(0,1)

Raven: CARD($\mathbb{R}$) = CARD(0,A) where A is as small as I want

Len: and that equals c. c is really the CARD of the? [CONTINUUM]

Cantor is looking for giants, for another transfinite number. Where did he look?

Jozzy: The 2nd dimension.

Len: How many years did Cantor think about this?

William: 3

For him to prove there does not exist a 1-1 correspondence, how would he do it? [PBC] But what did he in fact prove?!? CANTOR found a 1-1 correspondence to show CARD($\mathbb{R}^2$)=CARD($\mathbb{R}$) = c.

1.15.2 Class

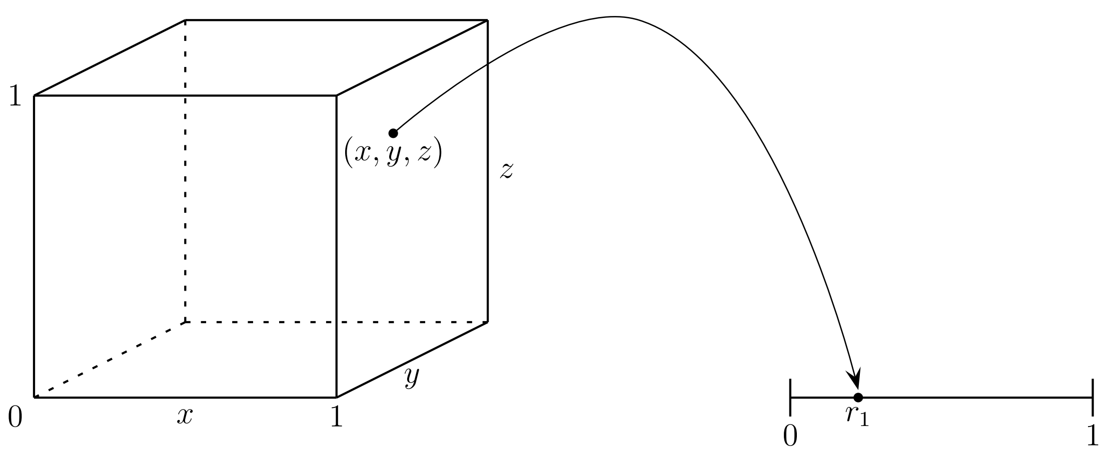
What question did Cantor look at next? (after the 1st and 2nd dimensions?)

Let's look at the 3rd dimension. What dimension do we live in? [3rd]

Question of the Day

What is the CARD of the entire universe? CARD($\mathbb{R}^3$) = c

Proof (Idea):



If we take an arbitrary point, how many coordinates determine that point? [3] So it will have an x-coordinate, a y-coordinate, and a z-coordinate. Stop: Am I being clear? Suppose

$$(x, y, z) = (0.123456..., 0.83838383..., 0.9876...)$$

Is x rational? Why? y? Cantor was sitting around, and who came to help him? [Soupy!]

Len: How are you going to map this? [Phone starts ringing, Len asks - Was that Soupy?!?] What was the first decimal?

$$r_1 = 0.[1][8][9][2]...$$

Raven = 1

Second? Jozzy = 8

Raekwon = 9

A bunch of people say OH!!!

Len: Who sees what Soupy is up to? Great, then in one voice, can you give me the next 3 numbers? Let's say

$$r_2 = 0.[123456789....]$$

Where do we put r_2 in the cube if we use the Soupy Shuffle? They get

$$x = 0.147$$
$$y = 0.258$$
$$y = 0.369$$

Copy that and as soon as you're down eyes up front. I love it - students keeping track of weirdness!

WEIRD FACT #8

CARD($\mathbb{R}^3$) = c

So getting poetic, CARD(Universe) = CARD(A Grain of Sand)

Guys - this is way more profound than you realize. Do you know what is the prevailing theory of the origins of the universe? [Big Bang Theory] Can someone address the class and explain the theory?

Donte does

Out of nothingness [emerges a star]

Singularity - started 13.5 billion years ago. It has been expanding ever since then from that point.

A re-statement of the Jamaulian Hypothesis? There exist only two transfinite numbers. Who thinks WEIRD FACT #8 proves this?

Jamaul: How can something be between countable and uncountable?

AMAZING: Cantor discovered a 3rd transfinite number! He couldn't look any further than the universe, so he looked inside sets.

Definition: A set $\mathcal{A}$ is a subset of a set $\mathcal{B}$ if every element of $\mathcal{A}$ [is in/is an element of $\mathcal{B}$]

Example: Let S = { All students }

Suppose CARD(S) = 43. Let's look at all 43 possible subsets of the class.

Example = { Julius }

{ Ivan, Julius }

How many subsets are there?