Math as Flexibility of Mind

Volume II: Roses

1 Course on Infinite Series

1.1 Day 1 (Week 1)

$2^5 =?$

Define out loud what this combination of numbers means - "two to the fifth power equals..."

Ask them, is $2^5 = 10$? Fight with them about why it is!

Then write down:

Proof: To show $2^5 = 32$?

Can someone run to the board and circle the base?

Write down

$$2 \times 2 \times 2 \times 2 \times 2$$

Define $\therefore$ symbol, maybe joke about therefive?

Then write $2^5 = 2 \times 2 \times 2 \times 2 \times 2, \therefore 2^5 = 32$ and go through and work out the answer/multiplication one by one.

Now let's make a power chart:

$$
\begin{aligned}
2^{10} &= \\
2^9 &= \\
2^8 &= \\
2^7 &= \\
2^6 &= \\
2^5 &= 32 \\
2^4 &= \\
2^3 &= \\
2^2 &= \\
2^1 &=
\end{aligned}
$$

Question: To go up the power chart, what do I do? Let's start with 2^6. Discuss. After that: Please, everyone try to do the rest up to 2^{10}, and when you are done please fold your arms so I know. "Omar [CI] is checking papers for me."

Course created by Leonard Boehm and recorded by Dionissi Aliprantis. This version is from June 2022.

What about going down the power chart? For 2^3, please put it on your fingers. What do you see?

So what is 2^0? This is a deep, ontological question we could spend a long time discussing and having fun with. Instead, we are just going to use some inductive reasoning to just agree we will define it as $2^0 \equiv 1$.
Then keep going, expanding the power chart to 2^{-1}, 2^{-2}, 2^{-3}, ... Who sees the secret? Anyone get the pattern? $2^{-10} = \frac{1}{1,024}$

1.1.1 Notation

Write this symbol ("Sigma") down

$$\Sigma$$

Write "summation" on the board. Can someone circle the first three letters? So we are talking about addition.
Point out that usually it comes along with some invisible words: "from" - "to" - "of": Have the class read out loud while you point to the appropriate symbol/letter/number:

$$\sum_{i=1}^{3} 2^{-i}$$

"The summation from i equals 1 to 3 of 2^{-i}"

Guess: How many numbers am I going to have to add? What is the upper limit? What are the terms filling in the blank here:

$$\sum_{i=1}^{3} 2^{-i} = \_\_\_ + \_\_\_ + \_\_\_$$

1.2 Day 2 (Week 1)

"Notebooks should be out."

Let's take 30 seconds to go over yesterday's notes, focusing on the power-charts. I need a hand: What is 2^5?

Who can tell the class what is 2^{-5}? 2^{-10}? Who can explain to the class how s/he got that? (Remember "My esteemed colleagues...") Also review the Σ notation. Often times Len would ask a CI to "choose who will tell us such and such..." and to "judge whether we are good, tell me if you don't see enough support."

So for the homework, show me:

$$\sum_{i=1}^{3} 2^{-i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \ ?$$

Show me on your hands, what is the denominator? Then, what is the numerator? Can someone explain at the board? Let's read, "Why do we do that?"

Took a timeout: "You all were showing tons of support, that was great! Thanks, I'm proud of you guys."

Transition: "Pencils down, eyes up front." What if I changed the upper limit to a 4? What does the next term become? Who wants to share with the class? When someone answers $\frac{1}{16}$, ask how to get that. Who has got a power form equal to $\frac{1}{16}$ (ie, 2^{-4}?). Read it out loud. Now, what if we change the upper limit to a 5? Is this problem even possible?

Working the room: "Is there enough support?" "Who thinks we can do this problem?" "What if we changed it to 100?" Could anyone figure out this problem during the week for $1,000?

Now: What if we changed it to ∞? (Timeout: Not going to stop all of the time, but all the other kids supported so and so, very happy to see that.) Have someone spell out ∞=INFINITY. It keeps going forever... so if I never stop adding? I could write $\ldots$

But that is a WEIRD problem! Why? Can anyone answer why? What if you pass off between computers every 5 years?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = ?$$

This is obviously a really hard problem. Whoever solved this problem must have been a genius!

That GENIUS = ISAAC NEWTON

Raise your hand if you know the subject he invented to solve this problem?

CALCULUS

"You might want to write this down." Anyone know what else he figured out?

1st BIG IDEA - PARTIAL SUMS

"Let's read with my finger."

Let's write the first partial sum as S_1.

$$S_1 = \frac{1}{2}$$
$$S_2 =$$
$$S_3 =$$

Recall, S_3 was your homework, show me with your hands, "When my finger touches the board."

If/when they say $\frac{1}{8}$, point out it is the "term," discuss partial *sums*.

Go into detail describing S_2. One step into infinity, you get how far? What about 2? 3? How far do you get?

Homework:

$$S_4 =$$
$$S_5 =$$
$$\vdots =$$

What happens if you go further and further?

1.3 Day 3 (Week 1)

"Good afternoon team."

"Good afternoon Mr. Boehm."

What is 2^5? 2^4? 2^3?

Pick up your pencils and let's revisit our problem from yesterday. What is Σ? [Eric was called on, but did not remember. Mr. Boehm's response was the say, ok, Eric, can you pick someone out to remind us what this symbol means?] Infinite means it goes on forever.

Does anyone remember how to read without the invisible words? [After many came close but not right: "I guess nobody can do it!" Then kept letting people try until someone got it. Then wrote it down and went over it with the class.]

SUMMATION

FROM

TO

OF

Now: Let's take some steps into infinity [Have them give you the numbers one by one, "What's the next one?"]:

$$\sum_{i=1}^{\infty} 2^{-i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \tag{1}$$

Let's read the three dots as

"FOREVER AND EVER AND EVER."

Let's read 1 out loud together.

Q: Why is this a WEIRD problem?

A: Because it might not have an answer.

Timeout: Praise for focus on student answering the question.

Revisit: It took a GENIUS to solve this, anyone remember who? ISAAC NEWTON.

Q: What was his 1st big idea?

A: Partial sums.

"Let me give everyone 2 minutes to copy down 1 in today's notes."

Q: Does anyone remember how to read this - S_1?

A: "S sub-one."

Review of the 1st BIG IDEA - PARTIAL SUMS

Newton realized - I can't do this whole problem! But aha - I can do part of it!

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

What happens now if I take three steps to into infinity? What do I add?

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

[Read out loud with finger pointing to terms.]

So if you take one step into infinity you get?

If you take two steps into infinity you get?

If you take three steps into infinity you get?

If you take four steps into infinity you get?

But infinity is a long way! Is there anyone in class who thinks they see a pattern? A secret about the partial sums? Can anyone come to the board for S_4? Can anyone explain the logic? ["Way to hang tough Isaiah." "OK, you got the denominator, now what about the numerator?"]

"Can you pick a good listener to repeat the end of what they just said to the class?"

Let's recheck S_4. Can someone get $\frac{1}{2}$ as $\frac{}{16}$? What about $\frac{1}{4}$? [Take about 10 minutes to go over in detail.]

"Looking for some signals." "Seeing some great support from..."

HOMEWORK: If you really see the pattern, what is S_{10}? S_N for any number N?

1.4 Day 4 (Week 2)

1.4.1 Getting the Kids Settled/Starting Up

$\longrightarrow$ Passes out power chart and says let's just concentrate on base 2, let's go through one or two times and get started.

$\longrightarrow$ "Good afternoon class."

$\longrightarrow$ "Good afternoon Mr. Boehm."

$\longrightarrow$ [To get them concentrating:] Hands on your head. Can you pick a number between 1 and 10? Multiply that by 2. Add 10. Divide that by 2. And now subtract the number you started with. What number did you get? $[\frac{2x+10}{2} - x]$ Can you show me on your hands?

Class: Can you get out your notes and spend 45 seconds going over them carefully?

1.4.2 Class

What is 2^9? 2^8? "Way to use your power chart."

"Pencils down and eyes up front." What is 2^{-1}? 2^{-2}? What is the next question? "Let's support Isaiah..." Until you get:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \tag{2}$$

But suppose we wanted to write 2 succinctly? What was $\sum$? What letter do we use as an index? What were the lower and upper limits? The argument?

A couple people try, and since we've been off for a couple of days, I'm going to put up the invisible words:

SUMMATION

FROM

TO

OF

Now, what if I change the upper limit to a 5? Let them say $\frac{1}{32}$ and ask: What power form corresponds to $\frac{1}{32}$?

Does anyone remember what S_5 is? ["S sub-5", a partial sum] Does anyone remember its denominator? Its numerator?

Now: What does

$$\sum_{i=1}^{\infty} 2^{-i}$$

mean? Does anyone remember who was the genius who solved this problem? [ISAAC NEWTON] Anyone remember what he invented? [CALCULUS] How did he solve this problem? Anyone remember his

1st BIG IDEA? [PARTIAL SUMS]

"Now is the time to pick up your pencils. Let's get this in today's notes." What happens if I take a first step into infinity? How far did Newton get when he took two steps into infinity? Go through term by term to S_5, reading one line together in unison [CI is walking around the room to check notes.]

PARTIAL SUMS

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} = ? \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\ S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \\ S_5 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32} \end{aligned}$$

Stop here: Who sees the secret? "In your notes, can you put a box around the partial sums?" What is the power form of the denominator, 32? "Xavier knows, so I'll choose Joseph." Have them give you

$$S_5 = \frac{31}{32} = \frac{2^5 - 1}{2^5}$$

What about the power form of S_4? [$\frac{2^4-1}{2^4}$] Who has got the power form that corresponds to 8? "I like your support!" Now, let's go to the top of the mountain. If I write S_N, where N can be any natural number? [After they give it to you,] "Put a box around this in your notes:"

$$S_N = \frac{2^N - 1}{2^N}$$

[Read out loud after one student reads.] In today's notes would you write the word "Example"?

Example: $S_8 =$ ____

If we use the secret formula, who can give me the denominator? Start with 2^N, then use power chart to give actual numbers. "Let's support Kyla's choice." "Nice support!" Now everyone do S_9. [Adults go around the room checking work.] One student goes up and puts the answer on the board. Newton takes 1 step to infinity, how far does he get? With 2 steps? With 3? With 8? 9?

1.4.3 Wrapping Up/Question for Tomorrow

What happens to those partial sums the further and further Newton goes to infinity? "Think about it in your mind." [Finger pointing to the board to give ideas.]

1.5 Day 5 (Week 2)

1.5.1 Getting the Kids Settled/Starting Up

When the kids came in they were quite rowdy. Mr. Boehm waited until *everyone* calmed down and got quiet before starting. Then he asked: "With our signals, can you tell me, do you think we showed our greatness when we came in today?"

"Good afternoon class."

"Good afternoon Mr. Boehm."

[The point of writing this: He has some 'rituals' to start class to get kids focused and in the right mindset. This can be the head problem "Pick a number between 1 and 10. Double it. add ten. Divide by 2. Subtract the original number. Everyone show me on their hands what they got." ($\frac{2x+10}{2} - x = 5$), or whatever, but the point is to start some habits and make it easier for the kids to focus and get settled.]

1.5.2 Class

If I write $\sum$ on the board, does anyone remember the index we've been using all summer? The lower limit? The upper limit that makes this problem almost impossible? The argument? Can anyone read this [now on the board] with the invisible words?

Why is it almost impossible? Can we have a good listener repeat that answer? Does anyone remember the GENIUS who solved this problem? [ISAAC NEWTON] What his his big idea?

1$^{\text{st}}$ BIG IDEA: PARTIAL SUMS

Does anyone remember the secret formula for S_N? Let's get that on the board.

$$S_N = \frac{2^N - 1}{2^N}$$

"Let's support him because he is looking in his notes, and that's what our notes are for." What happened when Newton took one step into infinity? What was the 4th partial sum? [A phone rings in someone's bag, some kids are distracted. One who is not is Jayla:] "Jayla I'm proud of you." Can someone come to the board and write - using the secret formula - S_7? Once on the board, "Let's sit up straight and read (following Len's finger):"

$$S_7 = \frac{2^7 - 1}{2^7} = \frac{128 - 1}{128} = \frac{127}{128}$$

Now, if you have some time in team time, you can teach this to your TAs.

What do the partial sums do? ["Please, address the class ['My ESTEEMED COLLEAGUES.']. DO NOT look at me!"] A kid answers: They get larger. But how large do they get? Get someone to say that they keep getting bigger and bigger. This leads into

NEWTON'S 2nd BIG IDEA: LIMIT

Let's consider an example: Suppose the speed limit is 80 MPH.

Q: How fast can you drive your car without going over the speed limit? Can someone give me a number that is close to 80? [Someone gives 75.] Can anyone get any closer? List out the numbers given by the class:

<u>LIMIT:</u>80 MPH

75

79

79.5

79.9

Who supports Isaiah's 79.9? Who says that is the closest we can get? Can anyone get any closer? [$79.\overline{9}$ given by the class.] So I would be writing 9s for how long?

A student asks: But how would you ever get to 80? Len gets very excited, and tells the kids:

"When someone asks a great question, they are doing what? <u>Thinking</u>!! The best thinkers of all time asked the best questions of all time!" So now, who agrees that we'll never get to 80? If the speed limit is 80, how close can we get?

Now I have some more invisible words for you.
LIMIT
AS
GOES TO
OF

$$\lim_{N\to\infty}$$

Go over slowly and have the class read with your finger on the board.

Then write on the board:

$$\lim_{N\to\infty}\frac{1}{2^N}$$

[Have one student read along with your finger on the board, then have the whole class.] What do you think that LIMIT is going to be? Let's write two columns [headed by N and $\frac{1}{2^N}$] and a number line just focused on the unit interval.

"Pick up your pencils..." Fill in with the class for $N = 1, 2, \ldots$ both the columns and the number line. Where on the number line do I put this? Stop me when I get there. In your notes, mark that on your number line. Is $\frac{1}{2}$ closer to 1 or 0? But that's only one step! What about the 2nd step? The 3rd and 4th steps? When doing this, have them make tick marks to divide the unit interval into halves, fourths, eighths, etc. Get to

$$\lim_{N\to\infty}\frac{1}{2^N} = 0$$

Have one student read, and then the whole class.

1.5.3 Wrapping Up/Question for Tomorrow

What limit did NEWTON have to solve? $\lim_{N\to\infty}(\cdot)$? [$\lim_{N\to\infty}(\frac{2^N-1}{2^N})$]

1.6 Day 6 (Week 2)

1.6.1 Getting the Kids Settled/Starting Up

Let's look at our notes from yesterday for about 30 seconds so we can move quickly today. Emphasize that today might just be our toughest material all camp, a very demanding day, especially in terms of note-taking. While the students are looking over their notes Mr.

Boehm goes around the room observing and supporting each of the students by name. "Joi I'm seeing some great notes."

Let's go over the invisible words - anyone have good notes from yesterday?

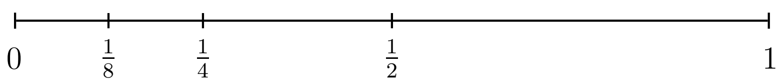
AS

GOES TO

OF

$$\lim_{N\to\infty} \frac{1}{2^N} =? \quad [0]$$

What happens as we take 1, 2, 3 steps into infinity? How close can we get? Read the invisible words again. Anyone recall NEWTON'S 2ND BIG IDEA? [LIMIT]



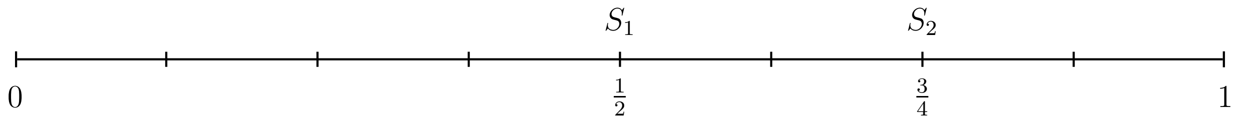
Now, make sure you have a big space in your notes, maybe start on the next page. We need really good notes today, college notes.

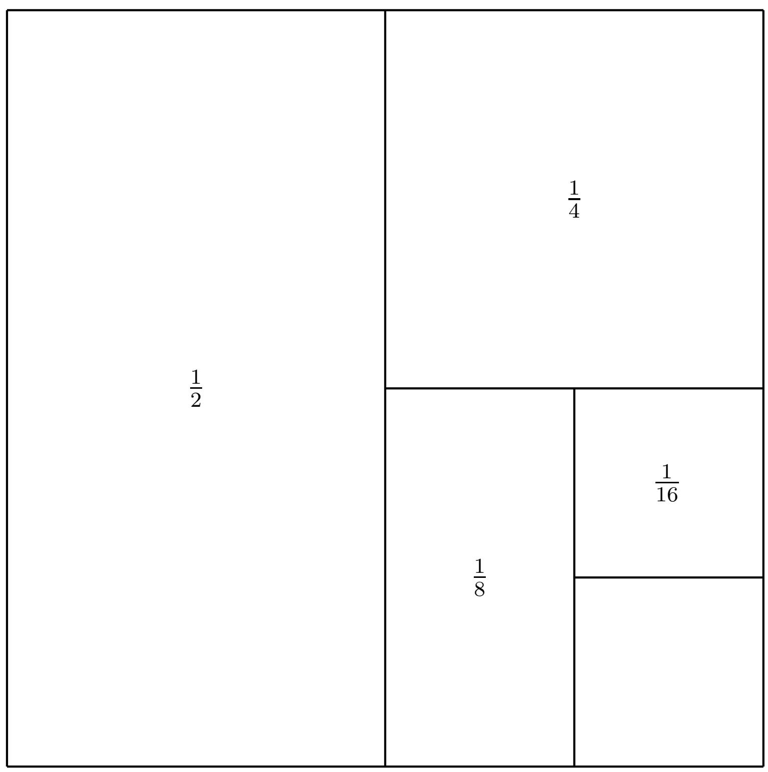
What was the problem that wasn't solved for thousands of years? Get them to put up $\sum_{i=1}^{\infty} 2^{-i}$. Why is this problem almost impossible? [Because it goes on forever and ever and ever...] Who was the genius who figured this out and what were their big ideas? Let's write out each step:

$$\sum_{i=1}^{\infty} 2^{-i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Do we ever stop adding? Let's draw a really long number line focused only on the unit interval. ["I see some great note-taking." CI's, feel free to check that notes are looking good.]

[Fill in the following figure piece by piece, slowly. Give them advice to put ticks halfway between two numbers.]





N	How close to 1 (Amount not eaten)
1	$\frac{1}{2}$
2	$\frac{1}{4} = ?\ [\frac{1}{2^2}]$
3	$\frac{1}{8}$
4	$\frac{1}{16}$
$\vdots$	$\vdots$
33	$\frac{1}{2^{33}}$

Mohammed and Joi - can you let me know when everyone is ready?

Now, let's start walking and eating pizza at the same time. Stop me when my finger gets to $\frac{1}{2}$ on the number line. "Mark that in your notes." Above $\frac{1}{2}$ write S_1. Divide the pizza in half, this is the part of the pizza we've eaten after one step. Use squiggles over the number line to represent the second column.

Now for 3, who remembers what is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$? Where does S_3 live on the number line? [Have someone come up and put $\frac{7}{8}$ on the board.] "All eyes on ..." Is $\frac{15}{16}$ closer to $\frac{7}{8}$ or 1? Discuss. If you take 4 steps to infinity, how close to 1 are you? Let's look at the pizza. Discuss. Who can give me the power form for $\frac{1}{4}$? Go through the right column.

Now, suppose we skip a whole bunch of steps and have taken 33 steps into infinity. How close to 1 are we? What is the power form? Who thinks that is pretty small? Anyone have a calculator? Can you tell me what it is?

$$\frac{1}{2^{33}} = 0.00000000011641532 \quad [1.1641532\mathrm{e}-10]$$

How many 0s are there?!? [9]

Let's look at the pizza. [Write out/draw many steps...] Hand: After 33 steps, how much of the pizza is left over? Based on this, what do you think NEWTON said was the answer to

$$\lim_{N\to\infty} S_N = ?$$

Do you ever actually get to 1? [No] But how close to 1 can you get? "Get this copied in your notes:"

$$\lim_{N\to\infty} S_N = 1 \Rightarrow \sum_{i=1}^{\infty} 2^{-i} = ?$$

This was Newton's GENIUS!!

1.6.2 Wrapping Up/Question for Tomorrow

You can always ask another interesting question. What problem do you think that NEWTON set out to tackle next?

$$\sum_{i=1}^{\infty} 3^{-i} = ??$$

[Some kids conjectured 2.]

1.7 Day 7 (Week 3)

1.7.1 Getting the Kids Settled/Starting Up

Start with the head problem ("Pick a number between one and ten...")

Let's spend 30 seconds going over our notes from last week. Anyone remember what this symbol "$\sum$" means? Anyone remember the lower limit we worked with? What was the upper limit that made this question almost impossible? The argument? [Get this on the board from their questions and get someone to read with your finger:]

$$\sum_{i=1}^{\infty} 2^{-i}$$

Why was this problem almost impossible? [It goes on forever and ever and ever...] Who was the genius who figured this out? [ISAAC NEWTON] Anyone remember what was his

1st big idea? [PARTIAL SUMS] What about the 2nd? [LIMIT] Read along w/ finger:

$$\begin{aligned}\sum_{i=1}^{\infty} 2^{-i} &= \lim_{N\to\infty} \overset{S_N}{\text{(PARTIAL SUMS)}} \\ &= \lim_{N\to\infty} \frac{2^N - 1}{2^N} \\ &= ?\text{[show me on your fingers} = 1]\end{aligned}$$

1.7.2 Class

[Today is a challenging notebook day. I think you guys are going to be able to write a calculus proof, but it's going to require really high-level, college-level notes.]

"One person who is writing fast, can you read for the class:"

$$\text{Proof:} \quad \sum_{i=1}^{\infty} 2^{-i} = ?$$

What were the first and second big ideas? Can anyone give me the secret formula? Let's support...

$$\text{Proof:} \quad \sum_{i=1}^{\infty} 2^{-i} = \lim_{N\to\infty} \frac{2^N - 1}{2^N} \tag{3}$$

"Eyes up front once you've got it down. Let's go for a team start..."

NEWTON's genius was to put these two BIG IDEAS together!!! Now let's look at an aside:

$$\frac{8-1}{8} = (?) - (?) = \frac{7}{8} \tag{4}$$

How could I write this as a subtraction problem? What two numbers, what two fractions would I need? [Len stops the class, I thought for being too rowdy. Instead, he says "Hey guys, I'm proud of you. That was cool Dorrian, you had your hand up, you knew the answer, and when I called on someone else you supported them."]

Read 4 with finger after kids get it. Now, let's rewrite 3 as one of two fractions [Have them give the answers, looking at the answer to 4 on the board.]:

$$\text{Proof:} \quad \sum_{i=1}^{\infty} 2^{-i} = \lim_{N\to\infty} \frac{2^N - 1}{2^N} = \lim_{N\to\infty} \left(\frac{2^N}{2^N} - \frac{1}{2^N} \right) \tag{5}$$

"Dr. Steve, do you see support for that?" In your notes, would you copy down the next step in this proof? Then can you sit with your arms crossed? Can someone read 5 out loud? Go over $\frac{2^N}{2^N}$ for $N = 1, 2, 3, \ldots$ What if I take 100 steps? So 3 is:

$$\text{Proof:} \quad \sum_{i=1}^{\infty} 2^{-i} = \lim_{N\to\infty} \frac{2^N - 1}{2^N} = \lim_{N\to\infty} \left(1 - \frac{1}{2^N} \right) = 1 - \lim_{N\to\infty} ? \tag{6}$$

Would you go off to the side and write $\frac{1}{2^N}$? What happens as N gets larger and larger and larger [You can use your power charts]?

What happens when you take 1 step? $[\frac{1}{2}]$

What happens when you take 2 steps? $[\frac{1}{4}]$

What happens when you take 3 steps? $[\frac{1}{8}]$

$[\frac{1}{16}]$

$[\frac{1}{32}]$

$[\frac{1}{64}]$

$[\frac{1}{128}]$

Now, what did Pops teach is in our course on the real numbers? The larger the denominator, what happens to the fraction? [Smaller pieces $\Rightarrow$ smaller numbers/fractions] Who says we can make the denominator as large as we want? $\Rightarrow$ We can make these fractions as ...? [small as we want] $\Rightarrow$ as $N \to \infty$ we go to what number? [Everybody sit up straight, I need you all.] How small can you make them? Everybody in your notes, please tell me what is this equal to?

$$\text{Proof:} \quad \sum_{i=1}^{\infty} 2^{-i} = \lim_{N\to\infty} \frac{2^N - 1}{2^N}$$
$$= \lim_{N\to\infty} \left(1 - \frac{1}{2^N}\right)$$
$$= 1 - \lim_{N\to\infty} \frac{1}{2^N}$$
$$= 1 - [0]$$

Let's make sure we get this in our notes. [write quickly] $\therefore$

$$\sum_{i=1}^{\infty} 2^{-i} = [1] \tag{7}$$

So we just proved what? "I see some great note-taking (with names)." What was the answer to NEWTON's question? Let's read 7 out loud.

Now let's go deeper and be a bit more precise. Instead of reading "equals" in 7, let's ask: Do these partial sums ever actually get to 1? The deeper way of saying 7 is the read the equal sign as "CONVERGES TO." What does that mean? [You can get as close as you want.] Let's read 7 using "CONVERGES TO."

What do you think the next argument is that we will look at?

$$\sum_{i=1}^{\infty} [3^{-i}] = ??$$

Who thinks this will converge? To what - can you show me on your hands? [Most students say 2.]

1.8 Day 8 (Week 3)

1.8.1 Getting the Kids Settled/Starting Up

"Good afternoon class."

"Good afternoon Dr. D."

Start with the head problem.

Can we go through our notes for 30 seconds? WHY do we do this? [Go fast, learn deeply. Circulate through the classroom.]

Can I get a volunteer to help me state the infinite series problem we have been studying?

Get your signals set [Get ready for PR]

$$[\sum_{i=1}^{\infty} 2^{-i}]$$

Anyone remember we had a word more specific that equals that we used when reading "=" for this problem? [CONVERGES TO] Can you show me on your hands what this converges to?

$$\sum_{i=1}^{\infty} 2^{-i} = [1] \iff$$

Anyone remember NEWTON'S TWO BIG IDEAS? Can you show me on your hands what this is?

$$\sum_{i=1}^{\infty} 2^{-i} = 1 \iff \lim_{N \to \infty} S_N = [1] \tag{8}$$

Why did NEWTON say this series converged to 1? Backup Q: What did we prove yesterday? Chorus reading of 8 with finger on the board. Have a student come to the board to give you

$$S_N =? \left[\frac{2^N - 1}{2^N}\right]$$

1.9 Class

What does "CONVERGES TO" mean? [Get closer and closer.] So if the LIMIT is 1, how close can we get? [As close as we want.]

Pencils down for a second. Question of the day: What would happen if I changed this argument to 3^{-i}? Christian hypothesizes that it converges to 2. Can someone say why they think that? Can a good listener repeat what they just said? BIG QUESTION: If that's the case, then what should this limit be? How would the secret formula change? Would you please write this in your notes:

$$\sum_{i=1}^{\infty} 2^{-i} \overset{?}{=} 2 \quad \Rightarrow \quad S_N \overset{?}{=} ? \quad \left[\frac{3^N - 1}{3^N}\right] \tag{9}$$

Chorus read 9. Kids, take out your power charts. Let's take some steps to infinity.

Have the kids fill in the partial sums if 9 were correct. "Every time you see an N, replace it with a 1."

$$N = 1 : S_1 = \frac{3^1 - 1}{3^1} = \frac{2}{3}$$

"Now, according to the secret formula, every time I see an N, replace it with a 2."

$$N = 2 : S_2 = \left[\frac{3^2 - 1}{3^2} = \frac{8}{9}\right]$$

Can someone volunteer to read the above expression? Then can we have a chorus reading? What about?

$$N = 3 : S_3 = ?$$

CIs circulate, have someone come to the board. Who sees the secret about this series?!? Prove it to me! Let's write up a table and sit up with your arms folded when you're ready. "Mr. Mulligan do you see enough folded arms?" Have a student come to the board to do $N = 5$:

N	$\left[\frac{3^N-1}{3^N}\right]$
1	$\left[\frac{2}{3}\right]$
2	$\left[\frac{8}{9}\right]$
3	$\left[\frac{26}{27}\right]$
4	$\left[\frac{80}{81}\right]$
5	$\left[\frac{242}{243}\right]$

Now let's stop and think for a second: Where do all these partial sums go? But how close to 1 can we get? What on this board can't be true?

$$\sum_{i=1}^{\infty} 3^{-i} = 2$$

Who can come to the board and write what they think this infinite sum is?

$$\sum_{i=1}^{\infty} 3^{-i} \stackrel{?}{=} [1]$$

Rihad conjectures: But they will all converge to 1! So you're saying:

$$\sum_{i=1}^{\infty} 4^{-i} \stackrel{?}{=} 1$$
$$\sum_{i=1}^{\infty} 5^{-i} \stackrel{?}{=} 1$$

etc? Please sit up straight and eyes up front - try to answer the toughest question: Anyone see something on the board that we are assuming is true but that might not be true? Any question marks on the board?

$$S_N \stackrel{?}{=} \frac{3^N - 1}{3^N}$$

Everyone get down this question for tomorrow: Is $S_N = \frac{3^N-1}{3^N}$ the true formula? [Have someone read out loud and then a chorus read.] Go home tonight and figure out:

$$S_1 \stackrel{?}{=} \frac{2}{3}$$
$$S_2 \stackrel{?}{=} \frac{8}{9}$$

1.10 Day 9 (Week 3)

"Good afternoon team."

"Good afternoon Mr. Boehm."

Let's go over yesterday's notes so today we can go in a hurry. Joi [CI], can you pick one student to read

$$\sum_{i=1}^{\infty}$$

What was the new argument we started looking at yesterday? [3^{-i}] Mo [CI], can you read this? I'll support you.

Let's take some steps into infinity, we can use our power charts to get [Reminder: these brackets mean to have the kids give you these answers]:

$$\sum_{i=1}^{\infty} = \left[\frac{1}{3}\right] + \left[\frac{1}{9}\right] + \left[\frac{1}{27}\right] + \left[\frac{1}{81}\right] + \left[\frac{1}{243}\right] + \cdots$$

Why is this a hard problem? [It goes on forever and ever and ever...] What is the other way of reading "="? [CONVERGES TO] What does that mean? [as close as you want] "In your notes, off to the side, would you make a chart. Somebody with great notes, what did we think the formula was from yesterday?" According to the formula, the 1st partial sum is? The 2nd? ... Get all these from the kids. Ask them to fold their arms when done.

N	$\left[\frac{3^N-1}{3^N}\right]$	S_N
1	$\left[\frac{2}{3}\right]$	
2	$\left[\frac{8}{9}\right]$	
3	$\left[\frac{26}{27}\right]$	
4	$\left[\frac{80}{81}\right]$	
5	$\left[\frac{242}{243}\right]$	

So we get closer and closer to what number? Can you show me on your finger? That is, the PARTIAL SUMS CONVERGE TO what number? [1] Now, get ready for a complex question. I would agree that it converges to 1, but only if something on the board were true. Could someone come to the board and circle the 1st term? So one step into infinity, $S_1 = \frac{1}{3}$. Who sees something that is false? What is the real partial sum? So based off of just one step - is this the right formula?

Let's figure out S_2:

$$S_1 = \frac{1}{3}$$
$$S_2 = \frac{1}{3} +? \left[\frac{1}{9}\right] =? \left[\frac{4}{9}\right]$$

How would we do this? [Find a common denominator.] Spend a bunch of time going over:

$$\begin{aligned}\frac{1}{3} + \frac{1}{9} &= \left(\frac{1}{3} \times \frac{3}{3}\right) + \frac{1}{9} \\ &= \frac{3}{9} + \frac{1}{9} \\ &= \frac{4}{9}\end{aligned}$$

[Chorus reading of all of the above.] Can someone come up and put up the 2nd partial sum? Get someone to give you $S_3 = \frac{13}{27}$. How did she get that so fact? [Discussion about banning impersonal pronouns, but rather using each other's names.] Dorrian says: Dividing by 2. Do I see agreement? For $\frac{40}{81}$?

Everyone in your notes, let's check this out. Is S_3 really $\frac{13}{27}$? Can someone check? Go through the steps again carefully, Len had Joseph teach the class.

Who wants to come to the board and show the real S_5? Who hasn't had a turn? She gets $\frac{111}{243}$. Team, let's support her. She works out $\frac{121}{243}$ after some work. Now, when you turn around, what do you see? [Agreement, her face lights up.] Now, can someone address the class and answer:

$$S_N =? \times \frac{3^N - 1}{3^N}$$

After someone gets it, "In today's notes, let's make sure we get this down:"

$$S_N = \frac{1}{2} \times \frac{3^N - 1}{3^N}$$

Chorus reading.

Last question for Monday: In your notes, put a star. What is

$$\lim_{N\to\infty}\left(\frac{1}{2}\times\frac{3^N-1}{3^N}\right) =??$$

Is this LIMIT > 1? $= 1$? < 1?

1.11 Day 10 (Week 4)

The kids come in and they are rowdy. Joi [CI] starts by encouraging the kids to get out their stuff. When Len comes in he waits for several minutes. Tells the kids: "I'm still waiting." Several minutes in he states that "We can cancel class or send some people out." He gets stern, he raises his voice, but he doesn't yell, and he doesn't get angry. Just serious. After saying everyone in the class has been busted, that we need greatness ALL the time, and noting that the class was starting 10 minutes late, he proceeds to name all of the kids who were ready and who were doing the right thing, praising them. (ends positive) I don't want to have to treat you guys like 7th graders, I want to treat you the way I treat my college students.

"Good afternoon team."

"Good afternoon Mr. Boehm."

Would everyone please quickly and quietly look at their notes from last time? If you don't have them, can you look over your power chart?

Mo, would you choose someone to read this part of the problem?

$$\sum_{i=1}^{\infty}$$

What was the argument we were looking at last time? [3^{-i}] Can someone read the whole thing?

$$\sum_{i=1}^{\infty}3^{-i}$$

Let's start to take some steps into infinity [kids give each term]:

$$\sum_{i=1}^{\infty}3^{-i} = \left[\frac{1}{3}\right]+\left[\frac{1}{9}\right]+\left[\frac{1}{27}\right]+\left[\frac{1}{81}\right]+\left[\frac{1}{243}\right]+\cdots \tag{10}$$

"Nice support Jayla." What does $\cdots$ mean? [You keep adding forever and ever and ever.] Once you have this down in today's notes, sit with your hands crossed so I know when to continue. Chorus reading of full expression in 10.

Notes to the kids that by not supporting, you are showing that you don't care about the person answering. The most important thing we do here is to support and care about each other. So let's do that.

What is the C-word? Does this series? [CONVERGE] When we say that an infinite series converges, who can explain that to me? Joseph explains, good but not completely perfect. Anyone want to embelish? Dorrian (girl) explains. What gets closer? What was NEWTON'S 1st BIG IDEA? Dorrian (boy) answers: [PARTIAL SUMS] If it converges to 1, can someone explain why? Someone says 0, and Len asks someone to come circle the first term, explains why one step in you are already past 0.

Eryn - I need you, please hang tough. Dorrian says that because the terms get smaller, the sum won't be larger than $\frac{1}{3}$. Len explains the differences between partial sums and terms. Writes out the partial sums under 10 and asks: What happens to these as we get closer and closer to infinity? I really appreciate the students sitting up straight. Who thinks it is $\frac{1}{2}$? Can someone explain? Please, address the class. All eyes on Jayla. Nice support Jasmine.

In the notes from last week there was a secret formula. Let's get a star up. Someone with really good notes from last week, can someone tell me from last Wednesday what is was? Kyla gets the start:

$$S_N = \frac{1}{2}\left[\frac{3^N - 1}{3^N}\right]$$

At this point someone interrupts again. Len stops to emphasize that we need to support each other and we need all eyes on the speaker. Two minutes later he calls another timeout to praise the exact same students for supporting the speaker.

Once you've got it down all eyes up front:

$$S_N = \frac{1}{2} \times \frac{3^N - 1}{3^N}$$

Now, let's verify, is this right? In today's notes, let's write:

CHECK:

$$\begin{aligned} N = 1 : S_1 &\left[= \frac{1}{2} \times \frac{3^1 - 1}{3^1}\right] \\ &\left[= \frac{1}{2} \times \frac{3 - 1}{3}\right] \\ &\left[= \frac{1}{3}\right] \end{aligned}$$

Let's get this copied and go for a team star. Chorus reading of all of the above expression. Does the secret formula check out for the 1st partial sum? Signals? So we take one step into infinity and we get? [$\frac{1}{3}$] Let's check out for 2 steps (reminds the students, any time you see an N, replace it with a 2):

$$N = 2 : S_2 \left[= \frac{1}{2} \times \frac{3^2 - 1}{3^2}\right]$$
$$\left[= \frac{1}{2} \times \frac{9 - 1}{9}\right]$$
$$\left[= \frac{4}{9}\right]$$

Who thinks this is getting easy? Now, the moment of truth: Does the secret formula give us the correct partial sum? Cross check the above expression with the partial sum written in 10.

OK, let's all try for $N = 3$.

$$N = 3 : S_3 =?$$

Len circulates, looks at notes. After a few minutes of letting the kids work out the problem, Len has Carla come up to the board and put up.

The question for tomorrow:

$$\lim_{N\to\infty} \left(\frac{1}{2} \times \frac{3^N - 1}{3^N}\right) =?$$

Let's get this copied quickly. Any conjectures about what this converges to? One student offers up $\frac{1}{2}$...

1.12 Day 11 (Week 4)

1.12.1 Review

We don't need to copy this in our notes for today. Can someone read like we would in a Calculus class?

$$\sum_{i=1}^{\infty}$$

What was the first argument we looked at? [2^{-i}] If we are going to be sophisticated, how are we going to read the equals sign? [CONVERGES TO] On your fingers, show Mo, what did we prove this converges to? [1] Now chorus read:

$$\sum_{i=1}^{\infty} 2^{-i}$$

Praises for support - even though you wanted to answer, you still were showing support. That was awesome. What does it mean to say that the series converges to 1? Dorrian explains it means you can get as close as you want. Len asks: What does? Joseph answers: the PARTIAL SUMS. What was the next argument we thought about? $[3^{-i}]$ What does

$$\sum_{i=1}^{\infty} 3^{-i}$$

converge to? Jasmyn says $\frac{1}{3}$. Jayla says $\frac{1}{2}$. Nice signals. Can someone explain why it might be $\frac{1}{2}$? I've got Rihad, I've got so and so, I've got... Jayla explains.

1.12.2 Class

Let's pick up our pencils and once you have this problem down please sit with your arms crossed so I know when to begin:

$$\sum_{i=1}^{\infty} 3^{-i}$$

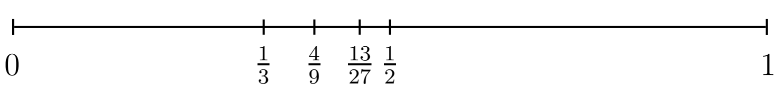
Let's tart taking some steps to infinity. Let's get out our power charts and a calculator.

I hate to stop class again, because we are in a hurry. But I need to for something very cool. Might not seem like a big thing, praises an act of kindness from Dorrian.

Have the kids give you the first couple of PARTIAL SUMS:

$$\sum_{i=1}^{\infty} 3^{-i} = \left[\frac{1}{3}\right] \left[+\frac{1}{9}\right] \left[+\frac{1}{27}\right] \left[+\frac{1}{81}\right] \left[+\frac{1}{243}\right] + \cdots$$

At the same time, draw a number line focused on the unit interval and a chart with N and S_N. Start filling it in with the kids help as you get the PARTIAL SUMS:



N	S_N
1	$\frac{1}{3} = [0.5\overline{0}]$
2	$\frac{4}{9} = [0.\overline{4}]$
3	$\frac{13}{27} = [0.\overline{481}]$

Some questions asked in our class:

Once we've taken one step into infinity, who says we are pretty close to infinity? 1? If we keep walking for the next ten minutes, will we get past $\frac{1}{2}$? Who says we will get right to $\frac{1}{2}$ Less than $\frac{1}{2}$? Someone with good notes from yesterday, what was S_2? Who has got a calculator and can write $\frac{4}{9}$ as a decimal? Way to hang tough Jayla. Wait! someone address the class: Is $\frac{4}{9}$ greater than or less than $\frac{1}{2}$? In your notes, draw a tick for $\frac{4}{9}$. What partial sum does that correspond to? [S_2] Anyone in your notes know S_3? Kyla hasn't had a turn... Who thinks, before we write it, is it greater or less than $\frac{1}{2}$? In your notes, put a tick mark for $\frac{13}{27}$ and write S_3. So what is going to happen as we go further and further into infinity? Xavier: the numbers are just going to get bigger and bigger. Len: How large can those PARTIAL SUMS get? Joseph: As close to $\frac{1}{2}$ as they want. Len: Can a good listener repeat that for me please?

Everybody sit up straight. I understand if you're not feeling well, but one person slouching affects all of us.

Who remembers from last time, what was the secret formula? Let's try for N=9 (Anyone have your power charts handy, what is 3^9?):

$$\begin{aligned} S_9 &= \left[\frac{1}{2} \times \frac{3^9 - 1}{3^9}\right] \\ &= \left[\frac{1}{2} \times \frac{19,682}{19,683}\right] \end{aligned}$$

Would everybody pick up their pencils and do $19,682 \div 2$? Based on what Chrishawna did, what is S_9? $\left[\frac{9,841}{19,683}\right]$ So go back and compute in your calculators, show me on your fingers: who thinks this is greater than $\frac{1}{2}$? Less than $\frac{1}{2}$? Equal to $\frac{1}{2}$?

Xavier: It is equal to 0.4999746...

Everybody take a close look at this. Everybody pick up your pencils and copy down your conclusion in your notes:

$$\therefore \lim_{N\to\infty} S_N = ?$$

Who wants to make a big statement? If we can get as close as we want, tell me when the chalk touches the board, what is this? [$\frac{1}{2}$]

Review that these are NEWTON'S 2 BIG IDEAS. NEWTON's GENIUS was to say this infinite series (ie, these partial sums) converge to?

$$\lim_{N\to\infty} S_N = \frac{1}{2} \iff \sum_{i=1}^{\infty} 3^{-i} = \frac{1}{2}$$

Can anyone read this? [Rihad does.] Then give a chorus reading for a team star: Mo and Joi [CIs]: You are the judge. Remember, when we read, we do so as one voice.

How many people think there's a secret about infinite series? If you're NEWTON, you've cracked two infinite series. What do you think would be the next infinite series he would look at? Dorrian answers:

$$\sum_{i=1}^{\infty} 4^{-i}$$

What do you think it converges to? Jayla answers $\frac{1}{4}$. Someone else explains the patten that as the base increases the sums decrease.

1.13 Day 12 (Week 4)

1.13.1 Review

"Good afternoon team."

"Good afternoon Mr. Boehm."

Pick up your pencils and write:

THE SECRET OF INFINITE SERIES

When someone asks you what we studied this summer you can tell them this. Now, anyone remember the secret? What was the first infinite series we studied?

$$\left[\sum_{i=1}^{\infty} 2^{-i}\right]$$

Show me on your fingers, what does this converge to? [1] Let's read together using "converges to":

$$\sum_{i=1}^{\infty} 2^{-i} = 1$$

What does CONVERGE mean? What gets closer? [PARTIAL SUMS] What was S_N? Have someone say and someone else read:

$$\left[\frac{2^N - 1}{2^N}\right]$$

Now, what was the second argument we looked at? [3^{-i}] Let's get the second infinite series copied and then look up so I know when do go on.

$$\left[\sum_{i=1}^{\infty} 3^{-i}\right]$$

What does that converge to?

$$\sum_{i=1}^{\infty} 3^{-i} = \left[\frac{1}{2}\right]$$

What about the partial sums? Have one student answer and another read:

$$S_N = \frac{1}{2} \times \frac{3^N - 1}{3^N}$$

1.13.2 Class

What was the next problem we were going to look at?

$$\left[\sum i = 1^{\infty} 4^{-i}\right]$$

Mo [CI], could you please tell the class the first time you saw this problem? [Towards the end of a calculus class in his first year of college.] So we showed the first series goes to 1, the second to $\frac{1}{2}$, so who thinks they see a secret? What is the above infinite series equal to? Chrishawna says 2. That's a good idea, any others? Dorrian says $\frac{1}{4}$. Discuss: Rihad explains that as the argument goes up, the sum goes down. So if Dorrian is correct, and we look at another infinite series, what do we think

$$\sum_{i=1}^{\infty} 5^{-i} = ?$$

Someone answers $\left[\frac{1}{8}\right]$. Why? That's great support! I want to give Chrishawna a star, she was agreeing, but before had a different answer, so it shows you are thinking. Now Joseph that was brilliant, but even more important was how you were caring about each other.

So if the series

$$\sum_{i=1}^{\infty} 4^{-i} = \frac{1}{4}, \tag{11}$$

that would mean what happens to partial sums? Let's take some steps into infinity: Let's look at our power charts:

$$\sum_{i=1}^{\infty} 4^{-i} = \left[\frac{1}{4}\right] \left[+\frac{1}{16}\right] \left[+\frac{1}{64}\right] \left[+\frac{1}{256}\right] \left[+\frac{1}{1,024}\right] + \cdots$$

When do I stop adding? The dots mean? [Keep adding forever and ever and ever...] Once you get this down, please sit with your arms crossed. Now in your notes skip a couple spaces and draw a number line. Let's focus on the numbers between 0 and 1.

If we take 1 step into infinity, what is S_1? Show me with a stop sign when I get there.

Anyone see something on the board they disagree about, can you point to it? Search for 11 and ask someone to explain why. If I keep going, who says I'll get past $\frac{1}{4}$? So $S_1 =$? [$\frac{1}{4}$] Let's put that on the number line. Now let's compute:

$$\begin{aligned} S_2 &= \left[\frac{1}{4} + \frac{1}{16}\right] \\ &= \left[\frac{1}{4} \times \frac{4}{4} + \frac{1}{16}\right] \\ &= \left[\frac{4}{16} + \frac{1}{16} = \frac{5}{16}\right] \end{aligned}$$

Hint: What has Pops been teaching you guys about Common Denominators (CDs)? Chorus read S_2 above once filled in. Joi [CI] disagrees that the class deserves a team star, perhaps motivated by the fact that she promised to buy them pizza if they got T^{10}. Fun stuff watching the kids' excitement over this. After putting $S_2 = \frac{5}{16}$ on the number line, Len asks: Who thinks this infinite series converges to $\frac{1}{4}$? Why? How many students changed their minds based on what they just saw? Dorrian notes that it is possible it goes to 1.

Now let's look at S_3:

$$\begin{aligned} S_3 &= \left[\frac{1}{4} + \frac{1}{16} + \frac{1}{64}\right] \\ &= \left[\frac{5}{16} + \frac{1}{64}\right] \\ &= \left[\frac{5}{16} \times \frac{4}{4} + \frac{1}{64}\right] \\ &= \left[\frac{20}{64} + \frac{1}{64} = \frac{21}{64}\right] \end{aligned}$$

Chorus read S_3 above once filled in. Dorrian (girl) points out that the secret formula is wrong. She suggests it is $S_N = \frac{4^N - 1}{4^N}$.

For Monday, what is the number we get closer and closer to? Homework problem for Monday, if you would be so kind. Please think about whether any of you see a secret about the partial sums:

$$S_4 = ?$$